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Physical Structure of the Energy-Momentum Tensor in General Relativity

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The algebraic classification of second-order symmetric tensors based on Segré type is used to give a systematic description of energy-momentum tensors in General Relativity. The uniqueness of the physical interpretation of a given energy-momentum tensor is discussed algebraically and a brief description of their "inheritance of symmetry" properties is also given.

1. INTRODUCTION

The purpose of this paper is to discuss a unified geometrical and algebraic approach to the study of the physical interpretation of the energy-momentum tensor in General Relativity. It will be shown that many simplifications result from a geometrical study of this tensor and its algebraic classification by means of Segré type and a brief review of this classification will be given at the end of this section. In Section 2, many standard energy-momentum tensors will be listed and classified, as will some (non-interacting) combinations of energy-momentum tensors. In Sections 3 and 4 the problem of the uniqueness of the physical interpretation of certain energy-momentum tensors will be discussed and some general results will be presented. Finally, in Section 5, the problem of the inheritance of metric symmetries by the energy-momentum tensor will be briefly considered. Some of the results presented have already appeared in the literature, but the approach followed here will be simpler and more systematic. Several new results will be presented and two errors in the literature will be corrected.

Throughout the paper, M will denote a space-time here assumed to be a smooth, four-dimensional, real manifold carrying a Lorentz metric g of signature $(-1, +1, +1, +1)$ whose coordinate components will always be

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denoted by g_{ab}. If $p \in M$, the tangent space to M at p will be denoted by $T_p(M)$. Latin indices take the value 0, 1, 2, 3, and round brackets will denote the usual symmetrization. In any particular coordinate system, the components of the Ricci tensor and Ricci scalar are denoted by R_{ab} and R, respectively, and Einstein's equations are

$$R_{ab} - \tfrac{1}{2}Rg_{ab} = 8\pi T_{ab} \tag{1}$$

where T_{ab} are the components of the energy-momentum tensor. For use later in the paper it is convenient to introduce a (real) null tetrad (l, n, x, y) at p, where $l, n, x, y \in T_p(M)$ and where the only nonvanishing inner products between tetrad members are $l^a n_a = x^a x_a = y^a y_a = 1$. Sometimes it is convenient to modify a null tetrad to a complex null tetrad $(l, n, m, \bar{m})$ where $\sqrt{2}\, m = x + iy$. These tetrads satisfy the following completeness relations:

$$g_{ab} = 2l_{(a}n_{b)} + x_a x_b + y_a y_b = 2l_{(a}n_{b)} + 2m_{(a}\bar{m}_{b)} \tag{2}$$

There are many approaches to the algebraic classification of the energy-momentum tensor in General Relativity (for reviews see Hall, 1983a, 1984a). For the present purposes, the classification by Segré type is perhaps the most convenient (Hall, 1976) (a brief review is given in Kramer et al., 1980). The Lorentz signature of the metric implies that the only possible Segré types (for $T_{ab} \neq 0$) are, in the Segré symbol notation, $\{1, 111\}$, $\{211\}$, $\{31\}$, and $\{z\bar{z}11\}$, where the digits inside the brackets refer to the multiplicity of the (real) eigenvalue represented and the $z\bar{z}$ pair in the fourth symbol refers to a pair of complex conjugate eigenvalues of multiplicity one. This last type is the only one where complex eigenvalues occur. Eigenvalue degeneracies will be denoted by enclosing the appropriate digits inside round brackets. The first type $\{1, 111\}$ is the diagonalizable (over $\mathbb{R}$) case and occurs if and only if T_{ab} has a timelike eigenvector. In this case, the first digit, separated from the others by a comma, represents the timelike eigenvalue. For each of the above types a null tetrad can be introduced at p such that the following canonical forms are, respectively, obtained:

$$T_{ab} = 2\sigma_0 l_{(a}n_{b)} + \sigma_1(l_a l_b + n_a n_b) + \sigma_2 x_a x_b + \sigma_3 y_a y_b \tag{3a}$$

$$T_{ab} = 2\sigma_1 l_{(a}n_{b)} + \lambda l_a l_b + \sigma_2 x_a x_b + \sigma_3 y_a y_b \tag{3b}$$

$$T_{ab} = 2\sigma_1 l_{(a}n_{b)} + 2l_{(a}x_{b)} + \sigma_1 x_a x_b + \sigma_2 y_a y_b \tag{3c}$$

$$T_{ab} = 2\sigma_0 l_{(a}n_{b)} + \sigma_1(l_a l_b - n_a n_b) + \sigma_2 x_a x_b + \sigma_3 y_a y_b \tag{3d}$$

where σ_0, σ_1, σ_2, $\sigma_3 \in \mathbb{R}$ and $\sigma_1 \neq 0$ in (3d). A detailed discussion of the eigenvector-eigenvalue structure of each type can be found in Hall (1976, 1983a, 1984a). Here, it is noted that complete sets of independent eigenvector (eigenvalue) pairs for the canonical forms in (3) are

$$l^a \pm n^a(\sigma_0 \pm \sigma_1),\ x^a(\sigma_2),\ y^a(\sigma_3) \text{ in (3a)}$$

$$l^a(\sigma_1),\ x^a(\sigma_2),\ y^a(\sigma_3) \text{ in (3b)}$$

$$l^a(\sigma_1),\ y^a(\sigma_2) \text{ in (3c)}$$

$$l^a \pm in^a(\sigma_0 \pm i\sigma_1),\ x^a(\sigma_2),\ y^a(\sigma_3) \text{ in (3d)}$$

An energy-momentum tensor satisfies the *dominant energy conditions* (Hawking and Ellis, 1973) if at each $p \in M$ and for each timelike vector $u \in T_p(M)$, $T_{ab}u^a u^b \geq 0$ and $T^a{}_b u^b$ is nonspacelike. These conditions forbid T_{ab} from having Segré type $\{31\}$ or $\{z\bar{z}11\}$ and restrict the other two cases by the following inequalities on the invariants σ_i: $\sigma_0 \leq 0$, $\sigma_1 \geq 0$, $\sigma_0 - \sigma_1 \leq \sigma_\alpha \leq \sigma_1 - \sigma_0$ $(\alpha = 2, 3)$ in (3a) and $\sigma_1 \leqslant 0$, $\sigma_1 \leq \sigma_\alpha \leq -\sigma_1$ $(\alpha = 2, 3)$ and $\lambda > 0$ in (3b) (Hall, 1983a, 1984a).

To close this section, it is remarked that each given Segré type together with a precise specification of its degeneracies determines a minimal polynomial relationship for the matrix T_a^b. Unfortunately the converse is not true, because, although the minimal polynomial determines the actual algebraic structure in the sense that it can distinguish among equations (3a)-(3d), it cannot in all cases determine the eigenvalue degeneracies. These minimal polynomial relationships are the "generalized Rainich conditions" (Hall, 1982, 1984a) because when applied to those energy-momentum tensors representing electromagnetic fields they yield the well-known Rainich conditions for these fields.

2. EXAMPLES

In this section, the algebraic structure of various (nonzero) energy-momentum tensors will be discussed. Since only algebraic properties are being considered, all energy-momentum tensors will be regarded as given at a particular point $p \in M$ in terms of a canonical tetrad at p.

2.1. Electrovac Space-Times

The algebraic structure of the energy-momentum tensor of an electrovac space-time is well known. One has

$$T_{ab} = \nu l_a l_b \qquad \text{(null case)} \tag{4}$$

$$T_{ab} = \mu(2l_{(a}n_{b)} - x_a x_b - y_a y_b) \qquad \text{(nonnull case)} \tag{5}$$

In (4) l is the repeated principal null direction of the null field and from (3b) the Segré type is $\{(211)\}$ with zero eigenvalue. In (5) (l, n, x, y) is a null tetrad with l, n the principal null directions of the field. The Segré type is $\{(1, 1)(11)\}$ with eigenvalues μ and $-\mu$ as follows from (3a). The dominant energy conditions give $\nu > 0$ in (4) and $\mu < 0$ in (5).

2.2. Fluid Space-Times

A viscous fluid with coefficient of dynamic viscosity η, bulk viscosity ζ, timelike fluid flow vector u^a chosen to be future-pointing and normalized so that $u^a u_a = -1$, energy density ρ (with respect to u^a), isotropic pressure p, shear tensor σ_{ab} $(=\sigma_{ba})$, expansion θ, and heat flow vector q^a has an energy-momentum tensor given by (see, for example, Misner et al., 1973)

$$T_{ab} = (p + \rho - \zeta\theta) u_a u_b + (p - \zeta\theta) g_{ab} - 2\eta\sigma_{ab} + 2u_{(a} q_{b)} \tag{6}$$

where $u^a q_a = \sigma^a_a = 0$ and $\sigma_{ab} u^b = 0$. It follows that the heat flow vector, if nonzero, is spacelike. One would normally impose the physical requirements $\eta \geq 0$, $\zeta \geq 0$ together with the dominant energy conditions (which imply that $\rho \geq 0$). However, it is of interest to note that the general form of equation (6), with no further conditions other than the contracted relations on u^a, q^a, and σ_{ab} immediately following it, imposes no algebraic restriction on T_{ab}. This follows because *any* second-order symmetric tensor can be decomposed into the general form (6). To see this, let S_{ab} be such a tensor at $p \in M$, u^a any unit timelike vector at p, and $h_{ab} = g_{ab} + u_a u_b$ the projection operator at p associated with u^a. Then

$$\begin{aligned} S_{ab} &= S^{cd}(h_{ac} - u_a u_c)(h_{bd} - u_b u_d) \\ &= (A + B) u_a u_b + B g_{ab} + \sigma'_{ab} + 2u_{(a} q'_{b)} \end{aligned} \tag{7}$$

where

$$\begin{aligned} &A = S^{cd} u_c u_d, \qquad B = \tfrac{1}{3} S^{cd} h^e{}_c h_{ed} \\ &\sigma'_{ab} = S^{cd} h_{ac} h_{bd} - B h_{ab} = \sigma'_{ba} \quad (\Rightarrow \sigma'_{ab} u^b = 0,\ \sigma'^a_a = 0) \\ &q'_a = -S^{cd} u_d h_{ca} \quad (\Rightarrow u^a q'_a = 0) \end{aligned} \tag{8}$$

For a viscous fluid with no heat flow ($q^a = 0$), equation (6) reveals that u^a is a timelike eigenvector of T_{ab} and so the Segré type is $\{1, 111\}$ or some degeneracy of this type. In the case of a perfect fluid ($\zeta = \eta = 0$, $q^a = 0$) the Segré type is $\{1, (111)\}$ with eigenvalues $-\rho$ and p and the dominant energy conditions are equivalent to

$$p - \rho \leq 0 \leq p + \rho \quad (\Rightarrow \rho > 0 \text{ if } T_{ab} \neq 0)$$

For a nonviscous fluid with nonzero heat flow one has $\zeta = \eta = 0$, $q^a \neq 0$ in (6). It then easily follows that u^a is not an eigenvector of T_{ab}, but with q^a, spans on invariant timelike 2-space of T_{ab}, and this rules out the possibility of T_{ab} having Segré type {31} or its degeneracy [for the definition and details of the invariant 2-space structure of T_{ab} see Hall (1983a, 1984a) and Cormack and Hall, 1979). Also, all members of the spacelike 2-space orthogonal to that spanned by u^a and q^a are eigenvectors of T_{ab} with eigenvalue p. Now let z^a be a unit spacelike vector parallel to q^a so that $q^a = qz^a$ $[q=(q^a q_a)^{1/2} > 0]$ and introduce any null tetrad (l', n', x, y) with $\sqrt{2}\, l'^a = z^a + u^a$ and $\sqrt{2}\, n'^a = z^a - u^a$. Then (6) yields

$$T_{ab} = [\tfrac{1}{2}(p+\rho)+q]l'_a l'_b + [\tfrac{1}{2}(p+\rho)-q]n'_a n'_b + (p-\rho)l'_{(a} n'_{b)} + p(x_a x_b + y_a y_b) \tag{9}$$

There are now three possibilities:

Case 1. $(p+\rho)^2 < 4q^2$. This is equivalent to $p+\rho-2q<0<p+\rho+2q$. A rescaling of l' and n' can then be used to cast (9) into the form (3d) and so one has the Segré type $\{z\bar{z}(11)\}$. Independent real eigenvectors are x^a and y^a with eigenvalue p and there are complex eigenvectors $l^a \pm in^a$ (which are easily written down in terms of the original field quantities) with corresponding eigenvalues $\frac{1}{2}(p-\rho) \pm \frac{1}{2}i[4q^2-(p+\rho)^2]^{1/2}$. This case would, of course, be ruled out by the energy conditions.

Case 2. $(p+\rho)^2 = 4q^2$. The dominant energy conditions and the condition $q>0$ show that here one must have $p+\rho-2q=0$. Equations (9) and (3b) then show that the Segré type is {2(11)} with no further degeneracies possible. Independent eigenvectors are l^a, x^a, and y^a with corresponding eigenvalues $\frac{1}{2}(p-\rho)$, p, and p. The dominant energy conditions imply that $p-\rho \le 0 < p+\rho$ and $3p \le \rho$, and so $\rho > 0$.

Case 3. $(p+\rho)^2 > 4q^2$. Here, the dominant energy conditions imply that $p+\rho \pm 2q$ are both positive. In this case a rescaling of l' and n' can be used to cast (9) into the form (3a) and the Segré type is {1, 1(11)}. The (unique) timelike eigendirection is spanned by $l-n$ and independent spacelike eigenvectors are $l+n$, x, and y with respective eigenvalues $-\rho+\varepsilon$, $p-\varepsilon$, p, and p, where

$$\varepsilon = \tfrac{1}{2}(p+\rho) - \tfrac{1}{2}[(p+\rho)^2 - 4q^2]^{1/2} > 0$$

(The eigenvectors $l \pm n$ are easily written in terms of the original field quantities.) The dominant energy conditions imply that $\varepsilon \le \rho - p$ and no further degeneracies in the above Segré type are possible.

2.3. Neutrino Fields

In this section *only*, the dominant energy conditions will be weakened to the single condition that at each $p \in M$, $T_{ab}u^a u^b \neq 0$ for each timelike vector $u \in T_p(M)$. For a neutrino field, the form of T_{ab} has been considered in Wainwright (1971) and Griffiths and Newing (1971) and will be discussed here only very briefly so as to bring it into the general scheme of the Segré classification. Here, a complex null tetrad $(l, n, m, \bar{m})$ exists such that, if the above weakened energy condition is satisfied, T_{ab} takes the form

$$T_{ab} = Al_a l_b - 2\phi\bar{\phi}\omega(4l_{(a}n_{b)} - g_{ab}) + 2i\phi\bar{\phi}(\bar{\sigma}m_a m_b - \sigma\bar{m}_a m_b) \qquad (10)$$

Here l is a geodesic null congruence whose twist ω and shear $|\sigma|$ satisfy $|\sigma|^2 - 4\omega^2 \leq 0$, A is real, ϕ complex ($\phi \neq 0$), and $A\omega \leq 0$. Consequently, $\omega = 0 \Rightarrow \sigma = 0$. One can convert (10) into the canonical forms given in (3) by writing $\sqrt{2}\, m^a = x^a + iy^a$ as in Section 1 and performing a spatial rotation in the xy plane to remove cross terms in x^a and y^a. The following possibilities arise (cf. Wainwright, 1971):

$$A \neq 0,\ \omega \neq 0,\ \sigma \neq 0,\ |\sigma|^2 \neq 4\omega^2 \rightarrow \text{Segré type } \{211\}$$

$$A \neq 0,\ \omega \neq 0,\ \sigma \neq 0,\ |\sigma|^2 = \omega^2 \rightarrow \{(21)1\}$$

$$A \neq 0,\ \omega \neq 0,\ \sigma = 0 \rightarrow \{2(11)\}$$

$$A \neq 0,\ \omega = 0,\ \sigma = 0 \rightarrow \{(211)\}$$

$$A = 0,\ \omega \neq 0,\ \sigma \neq 0,\ |\sigma|^2 \neq 4\omega^2 \rightarrow \{(1,1)11\}$$

$$A = 0,\ \omega \neq 0,\ \sigma \neq 0,\ |\sigma|^2 = 4\omega^2 \rightarrow \{1,11)1\}$$

$$A = 0,\ \omega \neq 0,\ \sigma = 0 \rightarrow \{(1,1)(11)\}$$

The eigenvectors and eigenvalues are easily obtained.

2.4. The Combination of Two Radiation Fields

This section deals with the combination of two (nonzero) noninteracting, radiation-type fields each with an energy-momentum tensor of the form (4) (one of which could be a null electromagnetic field). Let l' and n' be the null directions of these fields, so that the energy-momentum tensor becomes

$$T_{ab} = \nu_1 l'_a l'_b + \nu_2 n'_a n'_b \qquad (11)$$

It is assumed that l' and n' are nonparallel and scaled so that $l'_a n'^a = 1$. The dominant energy conditions for each field yield $\nu_1 > 0$, $\nu_2 > 0$. A scaling trick similar to that used in Section 2.2 then shows that T_{ab} has the form (3a) with Segré type $\{1, 1(11)\}$ and respective eigenvalues $-(\nu_1\nu_2)^{1/2}$, $(\nu_1\nu_2)^{1/2}$, 0, and 0. Clearly there are no further degeneracies and corresponding eigenvectors are easily computed in terms of the original field quantities.

2.5. The Combination of Two Perfect Fluids

Consider now the noninteracting combination of two (nonzero) perfect fluids where their (unit, timelike, future-pointing) flow vectors u and v are assumed nonparallel. If these fluids have densities ρ_α and pressures p_α $(\alpha = 1, 2)$, one has

$$T_{ab} = (p_1 + \rho_1)u_a u_b + (p_2 + \rho_2)v_a v_b + (p_1 + p_2)g_{ab} \tag{12}$$

It will be assumed not only that the separate dominant energy conditions hold, but that the extra conditions $p_\alpha \geq 0$ hold and so $p_\alpha + \rho_\alpha > 0$. One now chooses null vectors l' and n' in the timelike 2-space spanned by u and v such that $\sqrt{2}\, u^a = l'^a - n'^a (\Rightarrow l'_a n'^a = 1)$ and expresses v in terms of l' and n'. On substituting into (12) and rescaling l' and n' appropriately, as before, one obtains with the help of (2) the form (3a). The Segré type is $\{1, 1(11)\}$ with corresponding eigenvalues $-(\rho_1 + \rho_2) - \varepsilon$, $p_1 + p_2 + \varepsilon$, $p_1 + p_2$, and $p_1 + p_2$, where

$$\begin{aligned}2\varepsilon = &-(p_1 + p_2 + \rho_1 + \rho_2) + \{(p_1 + p_2 + \rho_1 + \rho_2)^2 \\ &-4(p_1 + \rho_1)(p_2 + \rho_2)[1 - (u_a v^a)^2]\}^{1/2}\end{aligned} \tag{13}$$

and so $\varepsilon > 0$. No further degeneracies are permitted and the eigenvectors are easily computed in terms of the original field quantities.

2.6. The Combination of a Radiation Field and a Perfect Fluid (Lichnerowicz, 1955; Goodinson and Newing, 1970)

For such a noninteracting combination of (nonzero) fields one has in an obvious notation

$$T_{ab} = \nu l'_a l'_b + (p + \rho)u_a u_b + p g_{ab} \tag{14}$$

The separate dominant energy conditions are again assumed, as is the extra condition $p \geq 0$, and so $p + \rho > 0$. The unit, timelike, future-pointing flow vector u of the fluid and the null direction l' of the radiation field determine a timelike 2-space and hence a null direction n' distinct from l' in this 2-space. The null vectors l' and n' are then chosen scaled such that $\sqrt{2}\, u^a = l'^a - n'^a$, which fixes ν. An appropriate scaling then leads to (3a) and Segré type $\{1, 1(11)\}$ with respective eigenvalues $-\rho - \varepsilon$, $p + \varepsilon$, p, and p where

$$2\varepsilon = -(p + \rho) + [(p + \rho)^2 + 2\nu(p + \rho)]^{1/2} > 0$$

No further degeneracies are possible and eigenvectors are determined as before.

2.7. The Combination of a Nonnull Maxwell Field and a Perfect Fluid (Lichnerowicz, 1955; Goodinson and Newing, 1970)

In the usual notation, one has

$$T_{ab} = \mu(2l_{(a}n_{b)} - x_a x_b - y_a y_b) + (p+\rho)u_a u_b + p g_{ab} \tag{15}$$

where the separate dominant energy conditions are assumed and also the extra condition $p \geq 0$. Hence $\mu < 0$ and $p+\rho > 0$. There are two cases to consider: the case where the principal null directions l and n of the electromagnetic field and the unit, timelike, future-pointing flow vector u are coplanar and the case when they are not.

In the coplanar case one can easily read off an orthonormal basis of eigenvectors of T_{ab} by inspection from (15). Alternatively, one can utilize the scaling freedom in l and n by choosing them according to $\sqrt{2}\,u^a = l^a - n^a$ and substituting into (15), using (2), to obtain (3a) and the Segré type $\{1, 1(11)\}$ with respective eigenvalues $\mu - \rho$, $\mu + p$, $-\mu + p$, and $-\mu + p$. No further degeneracies are possible and eigenvectors are determined as before. The unique timelike eigendirection is spanned by u.

In the noncoplanar case one notes that there is a unit spacelike vector x, determined up to a sign, and which is orthogonal to l, n, and u. It follows that x is an eigenvector of T_{ab} with eigenvalue $p - \mu$. Now construct the null tetrad (l, n, xy), which uniquely determines the unit spacelike vector y if one insists that $u^a y_a > 0$ (clearly $u^a y_a \neq 0$, since u, l, and n are independent). It is this null tetrad in which we regard the first term of (15) as being written. Then one has real numbers α, β, γ, δ such that

$$u^a = \alpha l^a + \beta n^a + \gamma x^a + \delta y^a \tag{16}$$

The conditions $u^a x_a = 0$ and $u^a y_a > 0$ imply that $\gamma = 0$ and $\delta > 0$. So $u^a - \delta y^a = \alpha l^a + \beta n^a$ and the timelike 2-spaces spanned by u and y and by l and n intersect nontrivially. However, they are not equal, because u, l, and n are independent. Hence their intersection determines a unique direction, a nonzero member of which may be written as $u'^a = u^a - \delta y^a$. Then $u'_a u'^a = -(1+\delta^2) < 0$ and so this direction is timelike. Now put $w^a = (1+\delta^2)^{-1/2} u'^a$, so that $w^a w_a = -1$. Also, w lies in the 2-space spanned by l and n and the scaling freedom in these null vectors can be used to choose them according to $\sqrt{2}\,w^a = l^a - n^a$. Now introduce along the pair of uniquely determined null directions in the timelike 2-space spanned by y and w null vectors p and q such that $\sqrt{2}\,p^a = y^a + w^a$, $\sqrt{2}\,q^a = y^a - w^a$. Since $x^a y_a = x^a w_a = 0$, one can introduce a null tetrad (p, q, x, z) which determines the unit spacelike vector z up to a sign. One now has

$$\sqrt{2}\,y^a = p^a + q^a, \qquad \sqrt{2}\,u^a = (\delta+\tau)p^a + (\delta-\tau)q^a \tag{17}$$

$$[\tau = (1+\delta^2)^{1/2}]$$

and the energy-momentum tensor can be readily converted to the null tetrad (p, q, x, z) by rewriting (15) using the completeness relations for both null tetrads (l, n, x, y) and (p, q, x, z). One writes

$$\begin{aligned} T_{ab} &= \mu(g_{ab} - 2x_a x_b - 2y_a y_b) + (p+\rho)u_a u_b + p g_{ab} \\ &= (\mu + p)(2p_{(a}q_{b)} + x_a x_b + z_a z_b) \\ &\quad - 2\mu(x_a x_b + y_a y_b) + (p+\rho)u_a u_b \end{aligned} \tag{18}$$

and then substitutes for y and u from (17) into (18) to obtain the required decomposition. The coefficients of $p_a p_b$ and $q_a q_b$ turn out to be positive and a simple scaling of p and q casts T_{ab} into the form (3a) with Segré type $\{1, 111\}$. The timelike eigenvalue is $-\rho + \mu - \varepsilon$ and the spacelike ones are $p - \mu + \varepsilon$, $p - \mu$, and $p + \mu$, where

$$2\varepsilon = [2\mu - (p+\rho)] + [(p+\rho-2\mu)^2 - 8\mu(p+\rho)(u^a y_a)^2]^{1/2} > 0 \tag{19}$$

Alternatively, one may replace $(u^a y_a)^2$ by

$$(u^a y_a)^2 = -(2\mu)^{-1}(E_{ab}u^a u^b + \mu)$$

where E_{ab} is the electromagnetic part of T_{ab}. With this notation, $-2\mu = (E_{ab}E^{ab})^{1/2}$. No degeneracies are possible. The eigenvectors are readily calculated in terms of the field quantities, with the eigenvector z having the especially simple form $\sqrt{2}\, z^a = l^a + n^a$ when the above choice of l and n is made.

2.8. The Combination of a (Nonelectromagnetic) Radiation Field and a Nonnull Maxwell Field

Here, one has

$$T_{ab} = \nu k_a k_b + \mu(2l_{(a}n_{b)} - x_a x_b - y_a y_b) \tag{20}$$

where k is the null direction associated with the radiation field, l and n are the principal null directions of the Maxwell field, and $\mu < 0$, $\nu > 0$. Again there are two cases, depending on whether k, l, and n are coplanar or not. In the coplanar case, k coincides with one of the null directions l or n and it easily follows that the Segré type is $\{2(11)\}$ (no further degeneracies) with independent eigendirections represented by k, x, and y and corresponding eigenvalues μ, $-\mu$, and $-\mu$. In the noncoplanar case, the procedure is very similar to that given in the noncoplanar case of Section 2.7 and need not be repeated. The Segré type turns out to be $\{1, 111\}$, with corresponding eigenvalues $\mu - \varepsilon$, $-\mu + \varepsilon$, $-\mu$, and $+\mu$, where $\varepsilon = \mu + (\mu^2 - \mu\nu)^{1/2} > 0$. No further degeneracies are possible.

To close this section, it is recalled that an observer is said to be "following the field (represented by an energy-momentum tensor T_{ab})" if he measures no energy-momentum flux across any spacelike 2-space orthogonal to his world line Pirani (1957). Such a condition obtains if and only if the timelike tangent vector to the observer's world line is an eigenvector of T_{ab}. Hence one can read off which of the energy-momentum tensors studied in this section admit such observers, since it is necessary and sufficient that they be of the Segré type $\{1, 111\}$ or some degeneracy of this type. In particular, the coplanar and noncoplanar cases in Section 2.7 are distinguished by the fact that in the former case, an observer moving along the fluid flow lines "follows the field."

3. UNIQUENESS PROBLEMS FOR FIXED FIELD TYPES

This section is concerned with the following question: if a certain energy-momentum tensor can be and is (algebraically) interpreted as a combination of energy momentum tensors of field types $A, B, \ldots, C$, are the particular members of $A, B, \ldots, C$ in the combination uniquely determined?

Clearly, if an energy-momentum tensor represents a perfect fluid with $p+\rho>0$, then it uniquely determines the fluid flow u^a and the quantities p and ρ. Similarly, if an energy-momentum tensor represents a null or a nonnull electrovac field, then the corresponding electromagnetic principal null directions are uniquely determined. The corresponding Maxwell bivector is then determined to within a duality rotation. These uniqueness results are trivial consequences of the algebraic structure of these particular energy-momentum tensors. In this section the uniqueness problem for more complicated types of field will be considered.

In the case of the combination of two radiation fields, T_{ab} can be written in the form (11) and uniquely determines the 2-space corresponding to the degenerate eigenvalue and hence its (timelike) orthogonal complement. This uniquely determines the pair of null directions l' and n' and so the null directions of the radiation fields are determined to within interchange.

Suppose now that one has the combination of two perfect fluids. Then the resulting energy-momentum tensor has the form (12) together with the restrictions given there. Are the physical characteristics of the fluid represented by the flow vector, the pressure, and the density uniquely determined? Suppose first that the flow vectors u and v are fixed. Since T_{ab} uniquely determines its eigenvalues, p_1+p_2 and $\rho_1+\rho_2$ are determined and $p_1+\rho_1$ and $p_2+\rho_2$ are determined to within an interchange of their values. Consequently, for each choice of the values of $p_1+\rho_1$ and $p_2+\rho_2$, there is a one-parameter family of values for the quadruple $(p_1, p_2, \rho_1, \rho_2)$ restricted, of course, by the requirements that $0 \leq p_\alpha \leq \rho_\alpha$ and $\rho_\alpha > 0$, $\alpha = 1, 2$. Now consider the possible freedom in the flow vectors u and v. The algebraic structure of T_{ab} shows that the timelike 2-space spanned by u and v is determined, and so if the same energy-momentum tensor also represents the combination of two other perfect fluids whose corresponding quantities are designated with primes, then $p_1 + p_2 = p'_1 + p'_2$, $\rho_1 + \rho_2 = \rho'_1 + \rho'_2$, and (12) gives

$$(p_1 + \rho_1)u_a u_b + (p_2 + \rho_2)v_a v_b = \beta u'_a u'_b + \gamma v'_a v'_b \tag{21}$$

where u, v, u', and v' are coplanar, and β and γ are positive and subject to $\beta + \gamma = p_1 + p_2 + \rho_1 + \rho_2$. Now rewrite the right-hand side of (21) in terms of $U_a = \sqrt{\beta}\, u'_a$ and $V_a = \sqrt{\gamma}\, v'_a$ and then express U and V in terms of u and v. On substituting into (21) one finds a one-parameter family of solutions for the pair U and V restricted by the requirement that U and V be timelike. With such a solution for U and V the coefficients β and γ are reintroduced as in (21) and decomposed as $\beta = p'_1 + \rho'_1$ and $\gamma = p'_2 + \rho'_2$ subject to $p'_1 + p'_2 = p_1 + p_2$, $\rho'_1 + \rho'_2 = \rho_1 + \rho_2$, and $0 \leq p'_\alpha \leq \rho'_\alpha$ and $\rho'_\alpha > 0$, which can be done with the one-parameter freedom discussed above.

Next consider the case of the combination of a radiation field and a perfect fluid. The energy-momentum tensor is given by (14). Because of the algebraic structure of T_{ab}, p and ρ are uniquely determined and if (14) can be rewritten with a different radiation direction–flow vector pair (n, v), then l, u, n, and v are coplanar, lying in a timelike 2-space. Thus, there are only two choices for the direction n and, clearly, $l = n$ implies $u = v$. The other choice for n leads to a unique v (clearly distinct from u). These are the only possibilities in this case.

If the energy-momentum tensor is the combination of a nonnull electromagnetic field and a perfect fluid and given by (15), then, in the coplanar case, T_{ab} uniquely determines μ, ρ, p, u^a, and the pair of null directions l and n. Thus the separate energy-momentum tensors are determined. In the noncoplanar case, again μ, p, and ρ are determined and one must effectively solve the equation obtained by equating (15) with a similar expression containing primed quantities, except that $\mu' = \mu$, $p' = p$, $\rho' = \rho$. With the help of the completeness relation (2) this reduces to

$$(p + \rho)u_a u_b + 4\mu l_{(a} n_{b)} = (p + \rho)u'_a u'_b + 4\mu l'_{(a} n'_{b)} \tag{22}$$

Now let x be the unit spacelike eigenvector of T_{ab} with eigenvalue $p - \mu$, determined to within a sign, and let y' be the vector for the primed fields corresponding to the vector y used in Section 2.7. Using the completeness relation (2) for the null tetrads (l, n, x, y) and (l', n', x', y') in (22) yields, after cancellation of the terms in $x_a x_b$ and g_{ab},

$$(p+\rho)u_a u_b - 2\mu y_a y_b = (p+\rho)u'_a u'_b - 2\mu y'_a y'_b (\equiv P_{ab}) \tag{23}$$

where $u^a y_a$ and $u'^a y'_a$ are both nonzero. Consideration of the invariant $p_{ab}p^{ab}$ shows that $(u^a y_a)^2 = (u'^a y'_a)^2$ and so, by changing the sign of y', if necessary, one can arrange that $u^a y_a = u'^a y'_a$. It turns out that there is essentially only one solution of (23) for the primed quantities other than the trivial one. To see this, note that the left-hand side of (23) has only two nonzero eigenvalues and they are distinct. This follows by a direct calculation from (23). These eigenvalues, say, κ_1 and κ_2, can be written down in terms of μ, p, ρ, and $u^a y_a$. The corresponding uniquely determined eigendirections can be represented by $k^1_a \equiv u_a + \alpha_1 y_a$ and $k^2_a \equiv u_a + \alpha_2 y_a$, where α_1 and α_2 are distinct and can also be written in terms of μ, p, ρ, and $u^a y_a$. Consequently, the right-hand side of (23) has eigenvalues κ_1 and κ_2 and eigendirections represented by $k'^1_a = u'_a + \alpha_1 y'_a$ and $k'^2_a = u'_a + \alpha_2 y'_a$. But $k^1_a k^{1a} = k'^1_a k'^{1a}$ and $k^2_a k^{2a} = k'^2_a k'^{2a}$. There are then four possibilities, $k^1 = \pm k'^1$ and $k^2 = \pm k'^2$, and these lead to two essentially different solutions of (23), the trivial one and the nontrivial one mentioned earlier. With this nontrivial solution introduced into the right-hand side of (23), one then subtracts $2\mu x_a x_b$ from both sides, completes the orthogonal spacelike pairs x, y and x', y' to null tetrads (l, n, x, y) and (l', n', x', y'), and, by use of the completeness relation (2), obtains (22). The resulting timelike 2-spaces spanned by the pairs l, n and l', n' are distinct, as are the timelike vectors u and u', and so the decomposition of T_{ab} into its electromagnetic and perfect fluid parts can be done in essentially two ways. In this respect, Goodinson and Newing (1970) seem to be in error.

The timelike, unit, future-pointing vectors u and u' and the corresponding y and y' are related by

$$u' = -\frac{\alpha^+ + \alpha^-}{\alpha^+ - \alpha^-} u - \frac{2\alpha^+\alpha^-}{\alpha^+ - \alpha^-} y$$

$$y' = \frac{2}{\alpha^+ - \alpha^-} u + \frac{\alpha^+ + \alpha^-}{\alpha^+ - \alpha^-} y \tag{24}$$

$$\alpha^\pm = \frac{(p+\rho-2\mu) \pm [(p+\rho-2\mu)^2 - 8\mu(p+\rho)(u^a y_a)^2]^{1/2}}{2(p+\rho)(u^a y_a)}$$

If the energy-momentum tensor is the combination of a nonnull electromagnetic field and a radiation field, the situation is essentially the same as that given above for a nonnull electromagnetic field and a perfect fluid in both the coplanar and noncoplanar cases, and so need not be discussed further.

For the energy-momentum tensor of a viscous fluid without heat flow, equation (6) holds with $q^a = 0$ and the flow vector u^a is a timelike eigenvector of T_{ab}. If T_{ab} has a unique timelike eigendirection, then u^a, ρ, and $p - \zeta\theta$ are uniquely determined. If, on the other hand, T_{ab} has more than one (and hence infinitely many) timelike eigendirections, it may be "projected" according to (7) and (8) with respect to any of these eigendirections to produce infinitely many different forms for the fluid (some of which may not satisfy the dominant energy conditions). However, all these timelike eigendirections have the same eigenvalue and so ρ and $p - \zeta\theta$ are still uniquely determined.

For a nonviscous fluid with heat flow, the dominant energy conditions imply that either $(p+\rho)^2 > 4q^2$ (Segré type $\{1, 1(11)\}$) or $(p+\rho)^2 = 4q^2$ (Segré type $\{2(11)\}$) (see Section 2.2). In the former case, similar arguments to those given above reveal that p, ρ, and q are determined, but that there are two choices for the pair u^a and q^a. In the latter case, p, ρ, q, u^a, and q^a are uniquely determined.

To close this section, it is remarked that although several of the single and combined energy-momentum tensors have the same Segré type, some distinction between them is possible by consideration of the size and numerical ordering of the eigenvalues.

4. THE ENERGY-MOMENTUM TENSOR AND ALGEBRAIC EQUIVALENCE PROBLEMS

There has been some recent discussion of the possibility that a given energy-momentum tensor may be interpreted physically in two quite different ways (Tupper, 1981, 1983; Raychaudhuri and Saha (1981, 1982). However, some of the algebra was lengthy and, in one case, an error was made. Without entering into the physics of the situation (that is, into the differential relations involved), a brief discussion of the algebraic problem will be given here. The results of the above authors will be rederived in an easier and more natural manner and some new results given. For the most part, only the general algebraic structure of the energy-momentum tensor will be considered, that is, no energy conditions will be imposed. The following results can now be given.

1. Let T_{ab} be a symmetric tensor at $p \in M$. Then:

(a) T_{ab} can be written in the algebraic form of a general fluid without heat flow if and only if T_{ab} admits a timelike eigenvector (equivalently, T_{ab} has Segré type $\{1, 111\}$ or some degeneracy of this type).

(b) if T_{ab} can be written in the algebraic form of a general fluid with nonzero heat flow, then this heat flow vector is an eigenvector of the shear tensor if and only if the fluid flow and heat flow vectors span a timelike invariant 2-space of T_{ab}.

The result (a) follows because if $q^a = 0$ in (6), clearly u^a is an eigenvector of T_{ab}. Conversely, if u^a is a timelike eigenvector of T_{ab}, then the projection equation (7) with respect to u^a holds and the last relation in (8) shows that the heat flow is zero. To prove (b) one contracts T_{ab} with u^a and q^a, respectively, and both implications follow immediately. In fact, if T_{ab} has a timelike invariant 2-space, then projecting T_{ab}, according to (7), with respect to any unit timelike member of it produces a corresponding heat flow vector which, if nonzero, lies in that 2-space and is an eigenvector of the associated shear tensor. The existence of a timelike invariant 2-space for T_{ab} rules out the possibility of it having Segré type {31} or its degeneracy.

2. If the energy-momentum tensor T_{ab} of an electrovac field is to be interpreted algebraically as that of a general fluid with zero heat flow, then the electromagnetic field is necessarily a nonnull field and the fluid flow vector must be a member of the (uniquely determined) timelike 2-space of eigenvectors of T_{ab}. The viscosity is nonzero and the shear tensor of the fluid has an eigenvalue degeneracy. If, on the other hand, T_{ab} is to be interpreted algebraically as that of a general fluid with nonzero heat flow, then either (a) the electromagnetic field is null, in which case there is no restriction on the fluid flow vector, which is necessarily coplanar with the resulting heat flow vector and the principal null direction of the electromagnetic field, the viscosity is nonzero, and the shear tensor of the fluid has an eigenvalue degeneracy, or (b) the electromagnetic field is nonnull, in which case the fluid flow vector must be outside the timelike 2-space of eigenvectors of the electromagnetic field and the viscosity is nonzero.

Most of the proof can be obtained by considering Segré types. The shear eigenvalue degeneracy is a direct consequence of the electromagnetic eigenvalue degeneracy, and the coplanar-ness referred to in (a) follows by equating the appropriate energy-momentum tensors and contracting with the fluid flow vector. In each case, the algebraic form of the fluid energy-momentum tensor can be obtained by an appropriate projection according to (7) and (8). The above result up to and including (a) was essentially contained in Tupper (1981) and Raychaudhuri and Saha (1981) and part (b) corrects an error in Tupper (1981).

3. If the energy-momentum tensor of a combined electromagnetic field and perfect fluid is to be algebraically interpreted as that of a general fluid without heat flow, then either (a) the electromagnetic field is null, in which case the general fluid flow vector is uniquely determined and is coplanar with the perfect fluid flow vector and the principal null direction of the electromagnetic field, the viscosity is nonzero, and the shear tensor has an eigenvalue degeneracy; or (b) the electromagnetic field is nonnull, with its

principal null directions coplanar with the perfect fluid flow vector, in which case the general fluid flow vector is uniquely determined, being parallel to the perfect fluid flow vector, the viscosity is nonzero, and the shear tensor has an eigenvalue degeneracy; or (c) the electromagnetic field is nonnull, with its principal null directions not coplanar with the perfect fluid flow vector, in which case the general fluid flow vector is uniquely determined and the viscosity is nonzero.

The proof follows easily by noting that the combination of an electromagnetic field and a perfect fluid leads to a unique timelike eigendirection. The general fluid energy-momentum tensor is obtained by projecting according to (7) and (8) in the obvious way. This theorem was originally given in Raychaudhuri and Saha (1982).

4. If the energy-momentum tensor of a combined electromagnetic field and a perfect fluid is to be algebraically interpreted as that of a general fluid with nonzero heat flow, then the general fluid flow vector can be in any timelike direction except the unique timelike eigendirection of the combined field. The viscosity is not zero if the electromagnetic field is null or if it is nonnull and not aligned with the perfect fluid.

The proof of the first part is clear. The final part is proved by noting that a general fluid with zero shear but nonzero heat flow leads to the Segré type $\{1,1(11)\}$ (which instantly rules out the nonnull, nonaligned case), where the repeated eigenvalue is the largest eigenvalue and is thus inconsistent with the null case (where the second largest eigenvalue is the repeated one).

5. If an energy-momentum tensor that is the combination of an electromagnetic field and a nonviscous fluid with nonzero heat flow has the algebraic form of a perfect fluid, then (a) the electromagnetic field must be null, (b) the repeated principal null direction of the electromagnetic field, the heat flow vector, and the two fluid flow vectors are coplanar, (c) the pressures and densities of the two fluids are equal, and (d) the two fluid flow vectors are nonparallel.

To prove this, let E_{ab}, T_{ab}, and T'_{ab} be the energy-momentum tensors of the electromagnetic field, the nonviscous fluid with heat flow, and the perfect fluid, respectively, so that, in the usual notation,

$$T'_{ab} - E_{ab} = T_{ab} \tag{25}$$

$$\begin{aligned} T'_{ab} &= (p'+\rho')v_a v_b + p' g_{ab} \\ E_{ab} &= \mu(2l_{(a}n_{b)} - x_a x_b - y_a y_b) \\ T_{ab} &= (p+\rho)u_a u_b + p g_{ab} + 2u_{(a}q_{b)} \end{aligned} \tag{26}$$

where $0 \le p \le \rho$, $0 \le p' \le \rho'$, $\rho > 0$, $\rho' > 0$, and $\mu < 0$. Now, the right-hand side of (25) has a degenerate spacelike eigenvalue, and so if we assume that the electromagnetic field is nonnull, it follows that the perfect fluid flow vector and the principal null directions of the electromagnetic field are coplanar. (This result requires a slightly different argument to that given in Section 2.7 because of the change of sign in (25). It is straightforward.) The uniquely determined degenerate 2-spaces on each side of (25) show that l, n, u, v, and q are coplanar and the facts that v is an eigenvector of the left-hand side of (25) and that u is not an eigenvector of the right-hand side show that u and v are not parallel. So write $v^a = f(u^a + gq^a)$, where f and g are nonzero, and substitute into the equation $E_{ab} = T'_{ab} - T_{ab}$, obtaining an expression for E_{ab} in terms of the two sets of fluid variables. The condition that q is an eigenvector of E_{ab} with eigenvalue μ $(=p-p')$ then yields $\mu > 0$, which is a contradiction. This establishes part (a) of the theorem. Now suppose that the electromagnetic field is null, so that, in the usual notation, $E_{ab} = \nu l_a l_b$, $\nu > 0$. Equation (25) then shows that l, v, u, and q are coplanar. This establishes the result (b). Comparing degenerate eigenvalues on either side of (25) and taking traces gives $p' = p$ and $\rho' = \rho$, which establishes result (c). The final result follows by assuming that u and v are parallel, substituting into (25), and contracting to obtain an easy contradiction. Theorem 5 was first given by Tupper (1983).

5. INHERITANCE OF SYMMETRY FOR THE ENERGY-MOMENTUM TENSOR

Suppose the metric g of the space-time M admits a one-parameter group of homothetic motions and let ξ be a vector field (homothetic Killing vector field) on M which generates it. One has, in components (and with the usual abuse of notation),

$$\mathscr{L}_\xi g_{ab} = \kappa g_{ab} \tag{27}$$

where κ is constant and $\mathscr{L}$ denotes the Lie derivative. In this section, a few remarks will be made concerning the "inheritance of symmetry" problem in General Relativity, namely the extent to which the symmetry embodied in (27) is inherited by the quantities that make up the source of the field represented by the energy-momentum tensor. This problem has been discussed by several authors (see Wainwright and Yaremovicz, 1976a, b, and the references contained therein). The situation when M admits a one-parameter group of motions (so that ξ is a Killing vector field) may be obtained from all results in this section by setting $\kappa = 0$. There is a good reason for considering homothetic motions, because in many ways they represent one of the most general types of symmetry transformations (McIntosh, 1979; Hall, 1983b, 1984b).

From (26) one easily finds $\mathscr{L}_\xi g^{ab} = -\kappa g^{ab}$ and it is then not difficult to show that $\mathscr{L}_\xi R^a_{bcd} = 0$ holds for the curvature tensor components. It then follows that $\mathscr{L}_\xi R_{ab} = 0$ and $\mathscr{L}_\xi R = -\kappa R$, and so, from (1),

$$\mathscr{L}_\xi T_{ab} = 0 \tag{28}$$

The following result summarizes the consequences of (27) and (28) for the algebraic structure of T_{ab}. A partial statement of it was given in Wainwright and Yaremovicz (1976a, b). The proof given here is of more general applicability (see Hall, 1985).

Suppose (27) and hence (28) holds. Then (a) if α is an eigenvalue of T_{ab}, $\mathscr{L}_\xi \alpha = -\kappa\alpha$, (b) if k is an eigenvector of T_{ab} corresponding to a nondegenerate eigenvalue, then $\mathscr{L}_\xi k^a = \beta k^a$ for some function β on M, and if k is nonnull, it may be scaled so that $\mathscr{L}_\xi k^a = -\frac{1}{2}\kappa k^a$ ($\mathscr{L}_\xi k_a = \frac{1}{2}\kappa k_a$).

To see this, let k be an eigenvector of T_{ab} with eigenvalue α, so that $T_{ab}k^b = \alpha g_{ab} k^b$ holds throughout M. Now let $p \in M$ and if $\xi(p) \neq 0$ consider the integral curve c of ξ through p. Let

$$\overset{p}{T}_{ab}, \qquad \overset{p}{k}{}^a, \qquad \overset{p}{g}_{ab}, \quad \text{and} \quad \overset{p}{\alpha}$$

be the values of the obvious quantities at p, so that

$$\overset{p}{T}_{ab}\overset{p}{k}{}^b - \overset{p}{\alpha}\overset{p}{g}_{ab}\overset{p}{k}{}^b = 0$$

holds at p. Then let T'_{ab}, k'^a, g'_{ab}, and α' be the quantities defined along c in some neighborhood of p by Lie dragging, along c, the corresponding quantities at p. Then (27) implies that $g'_{ab} = \sigma g_{ab}$, with $\sigma = e^{-\kappa t}$, and t an appropriate parameter along c, and (28) gives $T'_{ab} = T_{ab}$ and $\alpha' = \overset{p}{\alpha}$. It follows that $T_{ab}k'^b - \overset{p}{\alpha}\sigma g_{ab} k'^b$ has zero Lie derivative along c in the neighborhood considered and so is zero on c in that neighborhood, since it is zero at p. Thus the exact algebraic structure of T_{ab}, including degeneracies, is preserved as one moves along c with the eigenvalues being scaled by σ (since $\overset{p}{\alpha}\sigma = \alpha$) and so one has $\mathscr{L}_\xi \alpha = -\kappa\alpha$. This proves part (a) of the theorem. If α is a nondegenerate eigenvalue, it also follows that k'^a is parallel to k^a, and this leads to the statement (b) of the theorem after a scaling so that $k_a k^a = \pm 1$ in the nonnull case. The above argument still holds if α is a complex eigenvalue and k a complex (necessarily nonnull) eigenvector. The case when $\xi(p) = 0$ is straightforward.

It is remarked that this method, when applied to the algebraic Bel criteria (Bel, 1962), supplies details of the symmetry inheritance of the eigenvalues and principal null directions of the Weyl tensor.

A straightforward proof of this result is also available directly from definition if k is a nonnull eigenvector and α its corresponding eigenvalue. This can be extended to a complete proof by special consideration of the null eigenvectors in the Segré types {211} and {31}. However, the proof given above is shorter and of more general applicability. The result is also easily extended to the case where the eigenvalue α has exactly a double or a triple degeneracy, in which case the two- or three-dimensional distributions defined on M by the corresponding eigendirections of T_{ab} are preserved under Lie dragging.

The above results and the eigenvector–eigenvalue structure, together with degeneracies, for the fields and combinations of fields studied in previous sections now supply details of the symmetry inheritance for the energy-momentum tensor under homothetic motions. The case of a combined electromagnetic and perfect fluid field has been considered earlier (Wainwright and Yaremovicz, 1976a, b). In this case, for example, one finds $\mathscr{L}_\xi p = -\kappa p$, $\mathscr{L}_\xi = -\kappa \rho$, and $\mathscr{L}_\xi u_a = \frac{1}{2}\kappa u_a$ in both null and nonnull cases, together with $\mathscr{L}_\xi \mu = -\kappa\mu$ in the nonnull case. In both cases, the electromagnetic principal null direction(s) are Lie dragged parallel to themselves along c. It also easily follows that if the energy-momentum tensor has, at each $p \in M$, a unique, timelike eigendirection spanned by a future-pointing, unit, timelike vector u^a [as it must if it is to satisfy the dominant energy conditions stably (Hall, 1985)], then the expansion θ, the rotation w, and the shear σ of u^a satisfy $\mathscr{L}_\xi \theta = -\frac{1}{2}\kappa\theta$ and similar equations for w and σ.

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